Problem Set 6 – Relativity (`GOI36A`)

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The first half of this problem set is concerned with Killing vectors. Given a spacetime manifold, the Killing vectors on the manifold are associated with *isometries* of the spacetime, i.e. symmetries of the manifold that leave the metric invariant. A vector field K is a Killing vector if it satisfies the Killing equation:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \ .$$

As discussed in class, this condition guarantees that, in some suitable coordinate frame, K generates translations along one of the coordinate directions that leaves the metric invariant.

Killing vectors are an incredibly important topic in relativity, because in addition to telling us about isometries of the spacetime manifold itself, they also have a ton of useful and interesting properties that are worth understanding. So, we'll go ahead and work a lot with them on this problem set. We will also discuss Riemann normal coordinates in section 3, so that we can better understand what it means to go to a local inertial frame on any point in the spacetime.

1 Killing Vectors on S^2

I wanted to title this problem "Killing (Vectors) in the Name", but as your instructor I feel obligated to go more for clarity than pop culture references.

1.) Start with the familiar metric on S^2

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \ , \tag{1.1}$$

where we have set the radius of the sphere to one. Argue physically using the form of this metric that the vector $A = \partial_\phi$ should be a Killing vector. Then, prove this explicitly.

2.) Prove that the vector

$$C = -\sin\phi\, \partial_\theta - \cot\theta \cos\phi\, \partial_\phi \tag{1.2}$$

is also a Killing vector.

3.) We'll prove in a later problem that in general the commutator of any two Killing vectors is also a Killing vector. For now, just compute the vector B defined by

$$B = [C, A] \ , \tag{1.3}$$

where $[C, A] = CA - AC$ is the usual commutator. (Bonus: show explicitly that B is a Killing vector.)

4.) Show that the Killing vectors A, B, and C have the following commutation relations:

$$\begin{aligned} [A, B] &= C \ , \\ [B, C] &= A \ , \\ [C, A] &= B \ . \end{aligned} \tag{1.4}$$

This should be familiar; it is the Lie algebra of the rotation group $SO(3)$, the elements of which are all rotations about the origin of a 3D Euclidean space. Explain why it makes sense for the Killing vectors of S^2 to obey this algebra.

2 Various Fun Things to Prove about Killing Vectors

5.) Show that arbitrary linear combinations of Killing vectors are also Killing vectors.

6.) **Bonus:** given any two Killing vectors K_1 and K_2, show that their commutator $[K_1, K_2]$ is also a Killing vector. That is, if we define $K_3 \equiv [K_1, K_2]$, then you have to show that K_3 satisfies the Killing vector equation. You'll need to use the fact that

$$[\nabla_\rho, \nabla_\sigma]V_\mu = -R^\nu{}_{\mu\rho\sigma}V_\nu \ , \tag{2.1}$$

for any arbitrary vector field V.

7.) Suppose that you have a collection of matter fields in an arbitrary spacetime for which the corresponding stress-energy tensor $T^{\mu\nu}$ is conserved, i.e. $\nabla_\mu T^{\mu\nu} = 0$. And, assume K is a Killing vector. Then, if we define the current

$$J^\mu \equiv T^{\mu\nu}K_\nu \ , \tag{2.2}$$

prove that this current is conserved, i.e. $\nabla_\mu J^\mu = 0$.

8.) Consider a geodesic, parameterized by an affine parameter λ, in an arbitrary spacetime, and define $U^\mu = \frac{dx^\mu}{d\lambda}$. And, let K be a Killing vector in this spacetime. Show that along the geodesic, $K^\mu U_\mu$ is a constant.

3 Riemann Normal Coordinates

Consider an arbitrary Lorentzian spacetime manifold. We will not prove it, but it turns out that at any point P on the spacetime manifold, there exists a coordinate system such that the metric locally looks flat. By this, I mean that

$$\begin{aligned} g_{\mu\nu}(P) &= \eta_{\mu\nu} \ , \\ \partial_\rho g_{\mu\nu}(P) &= 0 \ . \end{aligned} \tag{3.1}$$

The coordinates that accomplish this are known as **Riemann normal coordinates**. They are incredibly useful because the relative simplicity of the metric makes many calculations simpler. In this problem, we will explore various aspects of Riemann normal coordinates.

9.) Once we are in such a Riemann normal coordinate frame, what happens to the Christoffel symbols $\Gamma^\rho_{\mu\nu}(P)$, evaluated at P? Can you say anything about $\partial_\sigma\Gamma^\rho_{\mu\nu}(P)$, i.e. derivatives of the Christoffel symbol?

10.) Use this result to show that the Riemann tensor, evaluated at P, is given by

$$R_{\mu\nu\rho\sigma}(P) = \frac{1}{2}\left(\partial_\sigma\partial_\mu g_{\nu\rho}(P) - \partial_\rho\partial_\mu g_{\nu\sigma}(P) + \partial_\rho\partial_\nu g_{\mu\sigma}(P) - \partial_\sigma\partial_\nu g_{\mu\rho}(P)\right) \ . \tag{3.2}$$

That is, the Riemann tensor is determined entirely by second derivatives of the metric at the point P. Importantly, the Riemann tensor is not necessarily zero! So, even though our metric looks locally like Minkowski space at the point P, the underlying manifold can still be curved and thus the Riemann tensor is non-trivial at the point P.

You should immediately appreciate how simple the expression (3.2) is, especially when compared to the fully general expression. This simplicity makes many properties of the Riemann tensor immediately obvious. For example, you should be able to easily see that the Riemann tensor satisfies the following (anti-)symmetry properties:

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu} \ . \tag{3.3}$$

Additionally, you can (with a just a little bit more work) prove that

$$R_{\mu[\nu\rho\sigma]} = 0 \ . \tag{3.4}$$

Of course, we have derived these properties in our nice Riemann normal coordinate system. However, these are all *covariant* expressions that involve only tensors. We can do coordinate transformations on these covariant expressions, and the result will be true in arbitrary coordinate frames. Thus Riemann normal coordinates provide a very easy way to prove the symmetry properties of the Riemann tensor.

11.) **Bonus:** use Riemann normal coordinates to prove the **Bianchi identity:**

$$\nabla_{[\lambda} R_{\mu\nu]\rho\sigma} \ , \tag{3.5}$$

where by square brackets we mean the fully antisymmetric combination of all indices inside the brackets.

Another reason to use Riemann normal coordinates is that the metric remains relatively simple within some small neighborhood of the point P. This can help us understand geodesics and distances, even on very complicated spacetimes. In the next few problems, we will see this simplicity explicitly.

12.) Without loss of generality, we can choose the point P to be at the coordinate $x^\mu = 0$. Show that any geodesic that goes through P must locally take the form

$$x^\mu(\lambda) = c^\mu \lambda \ , \tag{3.6}$$

for some set of constants c^μ.

13.) Now, differentiate the geodesic equation with respect to λ. Combine this new equation with the result (3.6) from problem 12 to show that the Christoffel symbols must satisfy

$$\partial_\nu \Gamma^\mu_{\rho\sigma}(P) + \partial_\rho \Gamma^\mu_{\sigma\nu}(P) + \partial_\sigma \Gamma^\mu_{\nu\rho}(P) = 0 \ . \tag{3.7}$$

14.) **Bonus:** prove that any number of (fully symmetrized) derivatives of the Christoffel symbol vanishes at P. That is, prove that

$$\partial_{(\nu_1}\partial_{\nu_2}\dots\partial_{\nu_n}\Gamma^\mu_{\rho\sigma)}(P) = 0 \ . \tag{3.8}$$

15.) Use the result (3.7) from problem 13 to prove that

$$\partial_\rho\partial_\sigma g_{\mu\nu}(P) = -\frac{1}{3}\left(R_{\mu\rho\nu\sigma}(P) + R_{\mu\sigma\nu\rho}(P)\right) \ . \tag{3.9}$$

Hint: as an intermediate step, you may find it useful to first prove that

$$\partial_\sigma \Gamma^\mu_{\nu\rho}(P) = -\frac{1}{3}\left({R^\mu}_{\rho\nu\sigma}(P) + {R^\mu}_{\nu\rho\sigma}(P)\right) \ . \tag{3.10}$$

16.) Finally, consider a new point P' that is at a coordinate x^μ in our Riemann normal coordinate frame. Do a Taylor expansion of the metric around the point P (which we set earlier to be at the origin $x^\mu = 0$) and then use the result (3.9) from problem 15 to show that

$$g_{\mu\nu}(P') = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\nu\sigma}(P) x^\rho x^\sigma + \mathcal{O}(x^3) \ , \tag{3.11}$$

where by $\mathcal{O}(x^3)$ we mean terms that contain at least three powers of the coordinate x^μ of the point P'.

As long as P' is sufficiently close to P such that we can ignore these higher-order terms, we have a very simple expression that tells us that the Riemann tensor is precisely related to how much the metric deviates from the Minkowski metric as we move away from P. So, even though we started with something that is locally flat at P, the underlying curvature of the manifold forces the metric to change as we move away from P.